

SYMPLECTIC RIGIDITY OF REAL BIDISC

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ABSTRACT. Let \mathbb{D} be the unit disc in \mathbb{C} , then $\mathbb{D}^n(r)$ is the complex or symplectic n -discs of radius r . Let $z_j = x_j + iy_j \in \mathbb{C}, j = 1, 2$ and $\mathbb{D}_{\mathbb{R}}^2 = \{(z_1, z_2) : |x_1|^2 + |x_2|^2 < 1, |y_1|^2 + |y_2|^2 < 1\}$ be the real bidisc. In this paper we will prove the following two theorems:

- (1) If $T \in O(4)$ is an orthogonal transformation on \mathbb{R}^4 , then $T(\mathbb{D}^2)$ is symplectomorphic to \mathbb{D}^2 w.r.t. the standard symplectic form on \mathbb{R}^4 if and only if T is unitary or conjugate to unitary.
- (2) For $r \geq 1$ and $n \geq 2$, $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ are not symplectomorphic w.r.t. the standard symplectic form on \mathbb{C}^n .

1. INTRODUCTION

Let $x_1, y_1, \dots, x_n, y_n$ be the standard coordinates on the $2n$ -dimensional Euclidean space $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the standard symplectic form on the space is given by $dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$. All symplectic embeddings considered in this paper will be with respect to the standard symplectic form, unless otherwise specified. Define the standard disc in \mathbb{C} of radius R by $\mathbb{D}(R) = \{z \in \mathbb{C} : |z| < R\}$, also define $\mathbb{D}_{\mathbb{R}}^2(r) = \{(z_1, z_2) \in \mathbb{C}^2 : |x_1|^2 + |x_2|^2 < r, |y_1|^2 + |y_2|^2 < r\}$ be the real bidisc of radius r . We denote $\mathbb{D}(1)$ by \mathbb{D} and $\mathbb{D}_{\mathbb{R}}^2(1)$ by $\mathbb{D}_{\mathbb{R}}^2$. We denote by $\mathbb{B}^{2n}(a)$ the $2n$ -dimensional Euclidean ball of radius a in \mathbb{R}^{2n} .

It is proved by Sukhov and Tumanov [7] that the real bi-disc $\mathbb{D}_{\mathbb{R}}^2$ cannot be symplectically embedded into the complex cylinder $\mathbb{D} \times \mathbb{C}$. If we consider the real bidisc as obtained from a non-holomorphic change of coordinates

$$T_0 : (x_1, y_1, x_2, y_2) \mapsto (x_1, x_2, y_1, y_2)$$

of \mathbb{D}^2 , then the result of Sukhov and Tumanov shows that $T_0(\mathbb{D}^2)$ is not symplectomorphic to \mathbb{D}^2 itself. The first main result of this paper generalizes this observation: if $T \in O(4)$ is any orthogonal transformation on $\mathbb{R}^4 = \mathbb{C}^2$, then $T(\mathbb{D}^2)$ is symplectomorphic to \mathbb{D}^2 if and only if T is unitary or conjugate to unitary. We will give a more precise statement in Section 3.

The second result of this paper considers a high dimensional analogy of the previous result. We will show that for $r \geq 1$ and $n \geq 2$, $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}(r)$ is not symplectomorphic to $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$.

The first striking result on symplectic rigidity was obtained by Gromov [3], which states that one can symplectically embed a sphere into a cylinder only if the radius of the sphere is less than or equal to the radius of the cylinder. Following Gromov's

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work, many results on symplectic rigidity were obtained for various domains. For example, McDuff [5] studied when a 4-dimensional ellipsoid can be symplectically embedded in a ball; Guth [4] gave an asymptotic result on when a polydisc $\mathbb{D}(r_1) \times \cdots \times \mathbb{D}(r_n)$ can be symplectically embedded into another. Our results have the same spirit, but we deal with essentially different domains: real bidisc and its modifications.

There are a lot of open problems concerning symplectic rigidity, for instance it is not known that whether $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}(r)$ is symplectomorphic to $\mathbb{D}^2 \times \mathbb{D}(r)$ when $r < 1$. The results in this paper only show that such symplectomorphism does not exist when $r \geq 1$. Another interesting open problem is to characterize when two given polydiscs are symplectomorphic.

2. J -HOLOMORPHIC DISCS AND SYMPLECTIC MANIFOLDS

In this section we will recall some basic properties of J -holomorphic discs and symplectic manifolds.

Definition 2.1. A smooth map $\phi : (M, J) \rightarrow (M', J')$ from one almost complex manifold to another is said to be (J, J') -holomorphic if its derivative $d\phi$ is complex linear, that is

$$(2.1) \quad d\phi \circ J = J' \circ d\phi.$$

Denote by J_{st} the standard complex structure of \mathbb{C}^n . A J -holomorphic disc or pseudo-holomorphic disc is a (J_{st}, J) -holomorphic map

$$u : \mathbb{D} \rightarrow M$$

from \mathbb{D} to an almost complex manifold (M, J) .

In local coordinates $z \in \mathbb{C}^n$, an almost complex structure J is represented by a \mathbb{R} -linear operator $J(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $J(z)^2 = -I$, where I is the identity map. Now the Cauchy-Riemann equations (2.1) for a J -holomorphic disc $z : \mathbb{D} \rightarrow \mathbb{C}^n$ can be written in the form

$$z_\eta = J(z)z_\xi, \zeta = \xi + i\eta \in \mathbb{D}.$$

We represent J by a complex $n \times n$ matrix function $A = A(z)$ and obtain the equivalent equations

$$(2.2) \quad z_{\bar{\zeta}} = A(z)\bar{z}_{\bar{\zeta}}, \zeta \in \mathbb{D}.$$

We recall the relation between J and A for fixed z . let $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a \mathbb{R} -linear map so that $\det(J_{\text{st}} + J) \neq 0$, where $J_{\text{st}}v = iv$. Set

$$Q = (J_{\text{st}} + J)^{-1}(J_{\text{st}} - J).$$

Lemma 2.2. ([1]) $J^2 = -I$ if and only if $QJ_{\text{st}} + J_{\text{st}}Q = 0$.

Notice that $QJ_{\text{st}} + J_{\text{st}}Q = 0$ is equivalent to Q being a complex anti-linear operator. Therefore Lemma 2.2 implies that there is a unique matrix $A \in \text{Mat}(n, \mathbb{C})$ such that

$$Av = Q\bar{v}, v \in \mathbb{C}^n.$$

Let M be a smooth manifold of real dimension $2n$. A closed non-degenerate exterior 2-form ω on M is called a symplectic form on M . A couple (M, ω) is

called a symplectic manifold. A basic example is $M = \mathbb{C}^n$ with the coordinates $z_j = x_j + iy_j, j = 1, \dots, n$. The form $\omega_{\text{st}} = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ is called the standard symplectic form on \mathbb{C}^n .

A symplectic form ω tames an almost complex structure J on M if $\omega(u, Ju) > 0$, for all $u \neq 0$. A basic example is $(M, \omega, J) = (\mathbb{C}^n, \omega_{\text{st}}, J_{\text{st}})$.

Lemma 2.3. ([1]) *Let J be an almost complex structure on \mathbb{C}^n , then J is tamed by ω_{st} if and only if the complex matrix A of J satisfies the condition*

$$(2.3) \quad \|A(z)\| < 1, \text{ for all } z \in \mathbb{C}^n.$$

Here the matrix norm is induced by the Euclidean inner product, that is, $\|A\| = \max_{0 \neq v \in \mathbb{R}^{2n}} |Av|_{\mathbb{R}^{2n}} / |v|_{\mathbb{R}^{2n}}$.

For a map $u : \mathbb{D} \rightarrow \mathbb{C}^n$, the (symplectic) area of u is given by

$$(2.4) \quad \text{Area}(u) = \int_{\mathbb{D}} u^* \omega_{\text{st}}.$$

If J is ω_{st} tamed, we can consider the canonical Riemannian metric $g_J(X, Y) = \frac{1}{2}(\omega_{\text{st}}(X, JY) + \omega_{\text{st}}(Y, JX))$ determined by J and ω_{st} . Suppose u is a J -holomorphic disc, then the symplectic area of u coincides with the area induced by g_J ; in particular, it coincides with the Euclidean area if $J = J_{\text{st}}$ (see [1] for more details).

3. ORTHOGONAL TRANSFORMATION OF COMPLEX BIDISC

Let $T \in O(4)$ be an orthogonal transformation on $\mathbb{R}^4 \cong \mathbb{C}^2$, let $\mathbb{D}^2 = \{|z_j| < 1 : j = 1, 2\}$ be the complex bidisc. In this section we will give a necessary and sufficient condition for $T(\mathbb{D}^2)$ to be symplectomorphic to \mathbb{D}^2 with respect to the standard symplectic form on \mathbb{C}^2 .

First of all, we define the notion of holomorphic radius and state a theorem proved by A. Sukhov and A. Tumanov [7] which provides a necessary condition on holomorphic radius for the existence of symplectic embedding.

Definition 3.1. Let Ω be a complex manifold. A closed set $A \subset \Omega$ is called a (complex) analytic set if it is, in a neighborhood of each of its points, the set of common zeros of a certain finite family of holomorphic functions. In this paper we only consider closed analytic sets.

Definition 3.2. A point p of an analytic set A in a complex manifold Ω is called regular if there is a neighborhood U in Ω containing p such that $A \cap U$ is a complex submanifold of U . The complex dimension of this submanifold is said to be the dimension of A at its regular point p , and is denoted by $\dim_p A$. The set of all regular points of A is denoted by $\text{reg} A$.

It is a fundamental result of complex analytic sets that the set of all regular points of an analytic set A is dense in A (see, for example, [2]).

Definition 3.3. A purely m -dimensional analytic set A is an analytic set such that for every $p \in \text{reg} A$, we have $\dim_p A = m$.

Definition 3.4. Let G be a domain in \mathbb{C}^n containing the origin. Denote by $\mathcal{O}_0^1(G)$ the set of closed complex purely one-dimensional analytic sets in G passing through

the origin. Denote by $E(X)$ the Euclidean area of $X \in \mathcal{O}_0^1(G)$. The holomorphic radius $\text{rh}(G)$ of G is defined as

$$\text{rh}(G) = \inf\{\lambda > 0 : \exists X \in \mathcal{O}_0^1(G), E(X) = \pi\lambda^2\}.$$

Example 3.5. Let $\mathbb{B}^4(r)$ be the Euclidean ball of \mathbb{C}^2 with radius r , then $\text{rh}(\mathbb{B}^4(r)) = r$. In fact the area $E(X)$ of $X \in \mathcal{O}_0^1(\mathbb{B}^4(r))$ is bounded from below by the area πr^2 of a section of the ball by a complex line through the origin (Lelong, 1950; see [2]).

The following theorem is known as Bishop's convergence theorem (see, for example, [2]), it will be used in the rest of the paper:

Theorem 3.6. *Let $\{A_j\}$ be a sequence of purely p -dimensional analytic subsets in a complex manifold Ω with locally uniformly bounded volumes:*

$$\text{Vol}_{2p}(A_j \cap K) \leq M_K < \infty$$

for any compact set $K \subset \Omega$. Here M_K is a constant depending only on K . Then we can extract a subsequence from $\{A_j\}$ converging on compact subsets in Ω (in Hausdorff sense) to a purely p -dimensional analytic subset or to the empty set.

The following result is due to A. Sukhov and A. Tumanov [7], it provides a necessary condition on holomorphic radius for the existence of symplectic embedding. This result will be used in the proof of Theorem 3.9.

Theorem 3.7. ([7]) *Let G_1 be a domain in \mathbb{C}^2 containing the origin and let G_2 be a domain in $\mathbb{D}(R) \times \mathbb{C}$ for some $R > 0$. Assume there exists a symplectomorphism $\phi : G_1 \rightarrow G_2$, then $\text{rh}(G_1) \leq R$.*

For $v = (v_1, \dots, v_4), w = (w_1, \dots, w_4) \in \mathbb{R}^4$, we denote the real inner product by $\langle v, w \rangle_{\mathbb{R}^4} = \sum_{j=1}^4 v_j w_j$. Similarly for $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{C}^2$, we denote the complex inner product by $\langle v, w \rangle_{\mathbb{C}^2} = \sum_{j=1}^2 v_j \overline{w_j}$. Notice that $\langle v, w \rangle_{\mathbb{R}^4} = \text{Re} \langle v, w \rangle_{\mathbb{C}^2}$.

By using the properties of inner product, the following lemma can be proved easily.

Lemma 3.8.

- (1) *Let $L \in \mathbb{C}^2$ be a real two dimensional plane. Denote by $L^{\perp_{\mathbb{R}^4}}$ the orthogonal complement of L with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ and by $L^{\perp_{\mathbb{C}^2}}$ the orthogonal complement of L with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$. If L is a complex line, that is $v \in L$ if and only if $iv \in L$ for all $v \in \mathbb{C}^2$, then $L^{\perp_{\mathbb{R}^4}} = L^{\perp_{\mathbb{C}^2}}$.*
- (2) *If $L \in \mathbb{C}^2$ is a complex line, then $L^{\perp_{\mathbb{C}^2}}$ is also a complex line.*

We denote by \mathfrak{J} the set consisting of four diagonal matrices:

$$\mathfrak{J} = \left\{ \begin{pmatrix} 1 & & & \\ & a & & \\ & & 1 & \\ & & & b \end{pmatrix} : a = \pm 1, b = \pm 1 \right\}$$

The following is the main theorem of this section. We used the canonical identification between complex matrices on \mathbb{C}^2 and real matrices on \mathbb{R}^4 :

Theorem 3.9. *Let $T \in O(4)$ be an orthogonal transformation. $T\mathbb{D}^2$ is symplectomorphic to \mathbb{D}^2 with respect to the standard symplectic form on \mathbb{R}^4 if and only if there exists $U \in U(2)$ such that $UT \in \mathfrak{J}$.*

Proof. (\Leftarrow) Suppose there exists an $U \in U(2)$ such that $UT \in \mathfrak{J}$, then we know that $UT\mathbb{D}^2 = \mathbb{D}^2$ as a set. Furthermore $U \in U(2)$ is a linear symplectomorphism on \mathbb{C}^2 . Hence $T\mathbb{D}^2$ is symplectomorphic to \mathbb{D}^2 .

(\Rightarrow) Let (z_1, z_2) be the coordinate on \mathbb{C}^2 . First of all, let $\partial\mathbb{D}^2 \cap \partial\mathbb{B}^4(1) = S_1 \cup S_2$ where $S_1 = \{|z_1| = 1, z_2 = 0\}$ and $S_2 = \{z_1 = 0, |z_2| = 1\}$. Therefore S_1 and S_2 are contained in the complex line $H_1 = \{z_2 = 0\}$ and $H_2 = \{z_1 = 0\}$ respectively. For $i = 1, 2$, let $u_i, v_i \in \mathbb{C}^2$ be orthonormal basis of TH_i under the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ on \mathbb{R}^4 . Note that TS_i can be parameterized by

$$\frac{1}{2} \left(t + \frac{1}{t} \right) u_i + \frac{1}{2i} \left(t - \frac{1}{t} \right) v_i$$

for $|t| = 1$ in \mathbb{C} . The complexification of TS_i , denoted by $\widetilde{TS_i}$, is given by the same parametrization but allowing $t \in \mathbb{CP}^1$. Here \mathbb{CP}^n is the complex projective space of complex dimension n . Hence $\widetilde{TS_i}$ is a complex algebraic curve in \mathbb{CP}^2 parameterized by $t \in \mathbb{CP}^1$.

Notice that for $i = 1, 2$, $\widetilde{TS_i}$ passes through the origin in \mathbb{C}^2 if and only if u_i and v_i are \mathbb{C} -dependent.

Suppose $T\mathbb{D}^2$ is symplectomorphic to \mathbb{D}^2 , then Theorem 3.7 implies that $\text{rh}(T\mathbb{D}^2) \leq 1$. By Theorem 3.6 there exists $X \in \mathcal{O}_0^1(T\mathbb{D}^2)$ such that $E(X) = \pi(\text{rh}(T\mathbb{D}^2))^2$. Suppose there exists $p \in \partial X \cap \partial\mathbb{B}^4(1)$ such that $p \in \text{Int}(T\mathbb{D}^2)$; then X is not entirely contained in $\mathbb{B}^4(1)$. Hence $E(X) > E(X \cap \mathbb{B}^4(1)) \geq \pi$ (see Example 3.5), which implies $\text{rh}(T\mathbb{D}^2) > 1$, a contradiction. Therefore $\partial X \subset \partial\mathbb{B}^4(1) \cap \partial T\mathbb{D}^2 = TS_1 \cup TS_2$ and X is a complex one dimensional analytic subset in $\mathbb{C}^2 \setminus (TS_1 \cup TS_2)$. Since $TS_1 \cup TS_2$ is a real one dimensional curve, it is totally real. Hence, by the reflection principle for analytic sets (see, for example, Section 20.5 of [2]), X extends as a complex one dimensional analytic set to a neighborhood of $TS_1 \cup TS_2$. By the uniqueness theorem X is contained in the complex algebraic curve $\widetilde{TS_1} \cup \widetilde{TS_2}$.

Since X contains the origin in \mathbb{C}^2 , without loss of generality we can assume $\widetilde{TS_1}$ contains the origin. By the discussion above, we know that u_1 and v_1 are \mathbb{C} -dependent. Hence $TH_1 = \text{span}_{\mathbb{R}}\{u_1, v_1\} = \text{span}_{\mathbb{R}}\{u_1, iu_1\} = \text{span}_{\mathbb{C}}\{u_1\} = \widetilde{TS_1}$. This shows that TH_1 is a complex line.

By Lemma 3.8, $H_2 = H_1^{\perp_{\mathbb{C}^2}} = H_1^{\perp_{\mathbb{R}^4}}$. Since T is an orthogonal matrix, we have $TH_2 = (TH_1)^{\perp_{\mathbb{R}^4}} = (TH_1)^{\perp_{\mathbb{C}^2}}$ where the last equality follows from Lemma 3.8 and the fact that $TH_1 = \widetilde{TS_1} = \text{span}_{\mathbb{C}}\{u_1\}$ is a complex line. Therefore Lemma 3.8 implies that TH_2 is a complex line.

We've shown that if T is orthogonal and $T\mathbb{D}^2$ is symplectomorphic to \mathbb{D}^2 , then T maps the complex lines $H_1 = \{z_2 = 0\}, H_2 = \{z_1 = 0\}$ to complex lines TH_1, TH_2 . Therefore there exists a unitary matrix $U \in U(2)$ such that $UT \in \mathfrak{J}$. \square

4. SYMPLECTIC RIGIDITY IN HIGH DIMENSIONAL CASE

Let $\mathbb{D}^m(r) = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_j| < r \text{ for } j = 1, \dots, m\}$ by the m -th product of discs of radius r . The following is the main theorem in this section:

Theorem 4.1. *For $r \geq 1$ and $n \geq 2$, the domains $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ in \mathbb{C}^n equipped with the standard symplectic form are not symplectomorphic.*

We will first give the proof for the case $r > 1$ by adapting the idea in the proof of Theorem 2.2 in [9]. We will then develop a new method to prove Theorem 4.1 for the case $r = 1$.

4.1. The case $r > 1$. In the case $r > 1$, theorem 4.1 follows from a more general result:

Theorem 4.2. *Given $r > 1$, for any real number $R > 0$, if there exists a symplectic embedding $\phi : \mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}(r) \rightarrow \mathbb{D}(R) \times \mathbb{C}^{n-1}$, then $R > 1$.*

Proof. For $R > 0$, suppose there exists a symplectic embedding from $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}(r)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$. It is proved in [9] that for every $1 \leq r_1 < \frac{2}{\sqrt{\pi}}$, there is a symplectic embedding from $\mathbb{B}^4(r_1)$ into $\mathbb{D}_{\mathbb{R}}^2$. Take $1 < r_1 < \frac{2}{\sqrt{\pi}}$, then by combining these two embeddings, we obtain an embedding from $\mathbb{B}^{2n}(a)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$ where $a = \min(r, r_1) > 1$. Therefore, by Gromov's non-squeezing theorem [3], we have $R > 1$. \square

4.2. The case $r = 1$. In order to prove the case $r = 1$, we need the following theorem regarding the existence of J -holomorphic discs, which is due to A. Sukhov and A. Tumanov [6]. The original statement was about the triangular cylinder $\Delta \times \mathbb{C}^{n-1}$ where $\Delta = \{z \in \mathbb{C} : 0 < \text{Im} z < 1 - |\text{Re} z|\}$ instead of the circular cylinder $\mathbb{D} \times \mathbb{C}^{n-1}$. However, one can see the result still holds for the circular cylinder by applying an area preserving map of the triangle to the disc.

Theorem 4.3. *(A. Sukhov and A. Tumanov [6]) Let A be a continuous $n \times n$ matrix function on \mathbb{C}^n with compact support in $\mathbb{D} \times \mathbb{C}^{n-1}$. Suppose there is a constant $0 < a < 1$ such that*

$$(4.1) \quad \|A(z)\| \leq a, \forall z \in \mathbb{D} \times \mathbb{C}^{n-1}.$$

Then there exists $p > 2$ such that for every point $x \in \mathbb{D} \times \mathbb{C}^{n-1}$ there is a solution $Z \in W^{1,p}(\mathbb{D})$ (Sobolev space) of equation (2.2)

$$Z_{\bar{\zeta}} = A(Z) \overline{Z_{\bar{\zeta}}}$$

such that $Z(\overline{\mathbb{D}}) \subset \overline{\mathbb{D} \times \mathbb{C}^{n-1}}$, $x \in Z(\mathbb{D})$, $\text{Area}(Z) = \pi$ and

$$Z(\partial\mathbb{D}) \subset \partial(\mathbb{D} \times \mathbb{C}^{n-1}) = (\partial\mathbb{D}) \times \mathbb{C}^{n-1}.$$

Furthermore, if we denote the components of Z by $Z = (f_1, \dots, f_n)$, then we have the following area property

$$\text{Area}(f_1) = \pi, \text{Area}(f_j) = 0, \text{ for } j = 2, \dots, n.$$

For $1 \leq j \leq n$, let M_j be the holomorphic disc $M_j = (m_1, \dots, m_n) : \mathbb{D} \rightarrow \mathbb{C}^n$ where $m_k(z) = 0$ if $k \neq j$ and $m_j(z) = z$. Notice that the minimal area of an analytic set passing through the origin in $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ is π , this is because $\mathbb{B}^{2n} \subset \mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ and the minimal area of analytic set of \mathbb{B}^{2n} passing through the origin is π (Lelong 1950; see [2]).

Lemma 4.4. *The minimal analytic set of $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ through the origin is given by one of the $n - 2$ distinct holomorphic discs M_3, \dots, M_n .*

Proof. Let $S_1 = \{x_1^2 + x_2^2 = 1, y_1 = y_2 = 0, z_3 = \dots = z_n = 0\}$, $S_2 = \{y_1^2 + y_2^2 = 1, x_1 = x_2 = 0, z_3 = \dots = z_n = 0\}$, $S_j = \{|z_j| = 1, z_k = 0 \text{ for } k \neq j\}$ for $3 \leq j \leq n$. By using Lelong's result (see [2]) and the argument in proof of Theorem 3.9, we conclude that the boundary of the analytic set E of minimal area in $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ through the origin must lie in the intersection of the boundary of \mathbb{B}^{2n} and the boundary of $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$, notice that this intersection consists of n circles S_1, \dots, S_n .

Suppose a boundary point of E lies in $S_1 \cup S_2$, then E must have a component lying in the complexification of $S_1 \cup S_2$, which is given by $\{z_1^2 + z_2^2 = 1, z_3 = \dots = z_n = 0\}$, in fact all of E lies in this set since E is of minimal area. However $\{z_1^2 + z_2^2 = 1, z_3 = \dots = z_n = 0\}$ does not pass through the origin, so the boundary of E is contained in the circles $S_3 \cup \dots \cup S_n$. Hence E is one of the discs M_3, \dots, M_n . \square

The following lemma is a consequence of Lemma 4.4 and Theorem 3.6:

Lemma 4.5. *Let E_j be a convergent sequence of analytic sets in $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ passing through the origin so that*

$$\lim_{j \rightarrow \infty} \text{Area}(E_j) = \pi.$$

Then the limiting analytic set E_{∞} is one of the $n - 2$ distinct holomorphic discs M_3, \dots, M_n .

Our proof of Theorem 4.1 in the case $r = 1$ is based on the fact that the domains $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ and \mathbb{D}^n have different number of analytic sets of minimum area through the origin. We are now ready to prove the main theorem in this section.

Theorem 4.6. *The domains $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ and \mathbb{D}^n equipped with the standard symplectic form on \mathbb{C}^n are not symplectomorphic.*

Proof. Suppose on the contrary that $\psi : \mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2} \rightarrow \mathbb{D}^n$ is a symplectomorphism. By composing a symplectomorphism of \mathbb{D}^n , we can assume that $\psi(0) = 0$.

Consider the standard almost complex structure J_{st} on $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ and let $J = \psi_* J_{\text{st}}$ be the complex structure on \mathbb{D}^n given by the push-forward of J_{st} by ψ . Since $\psi^* \omega_{\text{st}} = \omega_{\text{st}}$, the almost complex structure J is tamed by ω_{st} . Then the complex matrix \tilde{A} of J satisfies $\|\tilde{A}(z)\| < 1$ for $z \in \mathbb{D}^n$.

Let $\{K_l\}_{l=1}^{\infty}$ be a compact exhaustion of \mathbb{D}^n so that each K_l is a closed polydisc with radius less than 1, that is, $K_l \subset K_{l+1}$, K_l is a compact subset of \mathbb{D}^n for all l and $\cup_{l=1}^{\infty} K_l = \mathbb{D}^n$. For each l , let χ_l be a smooth cut-off function on \mathbb{C}^n with support in \mathbb{D}^n and equal to 1 on K_l . Define $A_l = \chi_l \tilde{A}$ to be a $n \times n$ matrix function on \mathbb{C}^n such that $A_l = 0$ outside \mathbb{D}^n . Since $\|\tilde{A}\| < 1$ on \mathbb{D}^n , there is a constant $0 < a < 1$ such that (4.1) holds for A_l . Let J_l be the almost complex structure on \mathbb{C}^n corresponding to the complex matrix A_l .

By considering \mathbb{D}^n as a subset of $\mathbb{D} \times \mathbb{C}^{n-1}$, we can apply Theorem 4.3 so that for each l , there exists a J_l -holomorphic disc $f_l : \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{C}^{n-1}$ such that the image of f_l passes through the origin. Also if we write $f_l = (f_{l,1}, \dots, f_{l,n})$, then we have $\text{Area}(f_{l,j}) = \delta_{j1} \pi$ for all l , here δ_{j1} is the Kronecker delta.

Fix an integer N , for each $l \geq N$, $\psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ is an analytic set in $\psi^{-1}(K_N) \subset \mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ passing through the origin. Since ψ is a symplectomorphism, we have

$$\text{Area}(\psi^{-1}(f_l(\mathbb{D}) \cap K_N)) \leq \text{Area}(f_l(\mathbb{D}) \cap K_N) \leq \pi.$$

Therefore by Theorem 3.6, after passing to a subsequence,

$$F_N = \lim_{l \rightarrow \infty} \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$$

exists and $\text{Area}(F_N) \leq \pi$. Notice that F_N is not an empty set for N sufficiently large, this is because $0 \in \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ for all $l \geq N$.

The above argument holds for all N , so we can apply Theorem 3.6 again to the sequence of analytic set F_N as $N \rightarrow \infty$. After passing to a subsequence, denote the

limit of F_N by F . Now F is an analytic set in $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ passing through the origin with $\text{Area}(F) \leq \pi$ and $\partial F \subset \partial(\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2})$. Since the minimal area of analytic set in $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ through the origin is π , so we must have $\text{Area}(F) = \pi$. Therefore F is one of the holomorphic discs M_j for $3 \leq j \leq n$ by Lemma 4.5.

Let $E = \psi(F)$. We now know that $\text{Area} f_l = \pi$ for all l and $f_l(\mathbb{D}) \cap \mathbb{D}^n \rightarrow E$ as $l \rightarrow \infty$, also we have $\text{Area}(E) = \pi$. We want to show that $f_l(\mathbb{D}) \rightarrow E$ as $l \rightarrow \infty$. Let $X_l = f_l(\mathbb{D}) \setminus \mathbb{D}^n$, that is the image of f_l which is not in \mathbb{D}^n . By the construction of A_l and J_l , we know that $J_l = J_{\text{st}}$ outside \mathbb{D}^n , hence X_l is an usual analytic set in $(\mathbb{D} \times \mathbb{C}^{n-1}) \setminus \mathbb{D}^n$. Since $\text{Area} X_l \leq \text{Area} f_l = \pi$ for all l , we can apply Theorem 3.6 to conclude that, after passing to a subsequence, X_l converges to an analytic set X . However $f_l(\mathbb{D}) \cap \mathbb{D}^n \rightarrow E$ as $l \rightarrow \infty$ and $\text{Area}(E) = \pi$ implies that

$$\lim_{l \rightarrow \infty} \text{Area}(f_l(\mathbb{D}) \cap \mathbb{D}^n) = \pi,$$

and by construction $\text{Area}(f_l) = \pi$ for all l , hence we have $\text{Area}(X_l) \rightarrow 0$ as $l \rightarrow \infty$. Therefore X is an empty set and we can conclude that

$$\lim_{l \rightarrow \infty} f_l(\mathbb{D}) \subset \mathbb{D}^n,$$

and hence

$$\lim_{l \rightarrow \infty} f_l(\mathbb{D}) = E.$$

Since $\text{Area}(f_{l,j}) = \delta_{j1}\pi$, if we write $\omega_{\text{st}} = \omega_1 + \cdots + \omega_n$ where $\omega_j = dx_j \wedge dy_j$ for $j = 1, \dots, n$, then we have

$$\int_E \omega_j = \delta_{j1}\pi.$$

Now for $1 \leq k \leq n$, by considering \mathbb{D}^n as a subset of the cylinder $\mathbb{C}^{k-1} \times \mathbb{D} \times \mathbb{C}^{n-k} \cong \mathbb{D} \times \mathbb{C}^{n-1}$, we can apply the above argument to obtain a real 2-dimensional set E_k in \mathbb{D}^n passing through the origin, satisfying the following conditions:

- (1) $\int_{E_k} \omega_j = \delta_{jk}\pi$ for $j = 1, \dots, n$, hence all E_k are distinct for $1 \leq k \leq n$.
- (2) The preimage $F_k = \psi^{-1}(E_k)$ is an analytic set in $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ passing through the origin.
- (3) F_k are distinct analytic sets for $1 \leq k \leq n$ since E_k 's are distinct and ψ is a bijection.
- (4) $\text{Area}(F_k) = \pi$ for $1 \leq k \leq n$.

Hence for each $1 \leq k \leq n$, F_k must be one of the holomorphic discs M_j for $3 \leq j \leq n$ according to Lemma 4.5, but this is impossible since all F_k 's are distinct, so we arrived at a contradiction. Therefore $\mathbb{D}_{\mathbb{R}}^2 \times \mathbb{D}^{n-2}$ and \mathbb{D}^n equipped with the standard symplectic form on \mathbb{C}^n are not symplectomorphic. \square

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